Ex1. Use an example from the Linear Algebra Review and Thm 1 to get a fundamental matrix for the system

$$\vec{Y}\,' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \cdot \vec{Y}$$

$\det \begin{pmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda) + 9$

$(2-\lambda)^2 = -9$

not possible, imaginary no.

using formula $\lambda_1, \lambda_2 = \dfrac{4 \pm 6i}{2}$

$\lambda_1 = 2+3i$

$\lambda_2 = 2-3i$

complex conjugate

$(a+bi)(a-bi)$

$a\lambda^2 + b\lambda + c = 0$

Eulers Formula $\left[e^{\pm xi} = \cos(x) \pm i\sin(x) \right.$

$i^2 = -1$

Find Eigenvectors

For $\lambda_1 = 2+3i$

$\begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \Rightarrow (-3iv_1 + 3v_2 = 0) \cdot -i = (-3v_1 + 3iv_2 = 0)$

Is it possible to get a **real-valued** fundamental matrix?

$-3iv_1 + 3v_2 = 0 \Rightarrow v_2 = iv_1 \Rightarrow \begin{pmatrix} 1 \\ i \end{pmatrix}$

For $\lambda_2 = 2-3i$

$\begin{bmatrix} 2-(2-3i) & 3 \\ -3 & 2-(2-3i) \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 3iv_1 + 3v_2 = 0 \\ -3v_1 + 3iv_2 = 0 \end{matrix}$

$\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Eigenpairs

$\left(2+3i, \begin{pmatrix} 1 \\ i \end{pmatrix}\right)$

$\left(2-3i, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right)$

$C(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{(2+3i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{(2-3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Euler's Formula

$e^{2t} e^{3it} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{2t} \left[(\cos(3t) + i\sin(3t)) \begin{pmatrix} 1 \\ i \end{pmatrix} \right] = e^{2t} \begin{pmatrix} \cos(3t) + i\sin(3t) \\ i\cos(3t) + i^2\sin(3t) \end{pmatrix}$

$= e^{2t} \left[\underbrace{\begin{pmatrix} \cos(3t) \\ -\sin(3t) \end{pmatrix}}_{\vec{y}_1(t)} + i \underbrace{\begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}}_{\vec{y}_2(t)} \right] \Rightarrow \vec{y}_1(t) = \begin{pmatrix} e^{2t}\cos(3t) \\ -e^{2t}\sin(3t) \end{pmatrix}$

$\vec{y}_2(t) = \begin{pmatrix} e^{2t}\sin(3t) \\ e^{2t}\cos(3t) \end{pmatrix}$

Fundamental set of real valued solutions

Real Valued general solution is

$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2$

$e^{\pm \pi i} = \cos(t) \pm j\sin(t)$

Important Remarks:

(i) Complex eigenvalues always occur in conjugated pairs. If $\vec{v}$ is an eigenvector associated to $\lambda = \alpha + \beta i$, then $\overline{\vec{v}}$ is an eigenvector associated to $\overline{\lambda} = \alpha - \beta i$.

(ii) The procedure given in the previous example always works for 2×2 and 3×3 matrices. That means, if λ is a complex eigenvalue and $\vec{v}$ is an eigenvector associated to λ, then the construction of the real fundamental matrix depends on the column vector

$$\mathbf{e}^{\lambda t}\vec{v}.$$

Ex2. Let A be a 2×2 matrix with real entries such that $A \cdot \begin{bmatrix} i \\ -1 \end{bmatrix} = (3+2i)\begin{bmatrix} i \\ -1 \end{bmatrix}$. Then, the real-valued solution of the i.v.p. $\vec{Y}\,' = A \cdot \vec{Y}$, $\vec{Y}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is:

a) $\mathbf{e}^{3t}\begin{bmatrix} 2\cos(2t) + 2\sin(2t) \\ 4\cos(2t) - 4\sin(2t) \end{bmatrix}$

b) $\mathbf{e}^{3t}\begin{bmatrix} 2\cos(2t) - 4\sin(2t) \\ 4\cos(2t) + 2\sin(2t) \end{bmatrix}$

c) $\mathbf{e}^{3t}\begin{bmatrix} 2\cos(2t) + 4\sin(2t) \\ 4\cos(2t) - 2\sin(2t) \end{bmatrix}$

d) $\mathbf{e}^{3t}\begin{bmatrix} 2\cos(2t) - 2\sin(2t) \\ 4\cos(2t) + 4\sin(2t) \end{bmatrix}$

$\vec{y}\,' = A\vec{y}$, A: 2×2

$A\vec{v} = \lambda\vec{v} \iff (\lambda, \vec{v})$ is an eigenpair

$$c(t) = e^{(3+2i)t}\begin{pmatrix} i \\ -1 \end{pmatrix} = e^{3t}e^{2it}\begin{pmatrix} i \\ -1 \end{pmatrix} = e^{3t}\underbrace{(\cos(2t) + i\sin(2t))}_{e^{2it}}\begin{pmatrix} i \\ -1 \end{pmatrix}$$

Euler's Formula

FOIL

$$= \begin{pmatrix} ie^{3t}\cos(2t) + i^2e^{3t}\sin(2t) \\ -e^{3t}\cos(2t) - ie^{3t}\sin(2t) \end{pmatrix} = \underbrace{\begin{pmatrix} -e^{3t}\sin(2t) \\ -e^{3t}\cos(2t) \end{pmatrix}}_{\vec{y}_1(t)} + i\underbrace{\begin{pmatrix} e^{3t}\cos(2t) \\ -e^{3t}\sin(2t) \end{pmatrix}}_{\vec{y}(t)}$$

General Real Valued Solution

$\vec{y}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t)$

$\vec{y}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = c_1\begin{pmatrix} 0 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix}$

$c_2 = 2 \qquad c_1 = -4$